



Jeffrey, M. (2017). An Update on that Singularity. In *Extended Abstracts Spring 2016: Nonsmooth Dynamics* (pp. 107-122). (Trends in Mathematics; Vol. 8). Birkhäuser Basel. https://doi.org/10.1007/978-3-319-55642-0_19

Peer reviewed version

Link to published version (if available):
[10.1007/978-3-319-55642-0_19](https://doi.org/10.1007/978-3-319-55642-0_19)

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An update on *that* singularity

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A bridge over troubled flows

Nothing epitomizes the intrigue of piecewise-smooth dynamics like the two-fold singularity. It is incredibly simple to describe — a point where a flow is tangent to a discontinuity threshold from both sides — yet intricate in its dynamics. Its understanding has pushed the boundaries of understanding in piecewise-smooth systems more than any other discontinuity-induced phenomenon.

It took nearly 30 years, from the translation of Filippov’s seminal book introducing the two-fold to the english speaking world, to resolving its switching layer behaviour, before we could say that the two-fold singularity was understood. And it *is* now understood, in wonderful detail: its structural and asymptotic stability [11, 9], its bifurcations including its local form and the affect of higher orders [11, 3, 4], the winding numbers when a flows rotates around it [6], the determinacy or determinacy-breaking that occurs when a flow passes through it [8], even its extension to multiple switches [10].

We now know that the two-fold singularity’s structural stability requires nonlinear switching or *hidden* terms, and that it comes in three main flavours, with numerous subclasses between which bifurcations can occur. We know that it is neither an attractor nor a repeller, but an organizing centre, a bi-directional bridge between attracting and repelling sliding on a switching surface, which can lead to the creation of a determinacy-breaking attractor (described as *non-deterministic chaos* in [3, 8, 1, 2]).

The developments towards understanding the two-fold singularity can be traced through the papers [7, 16, 11, 3, 4, 6, 9, 10]. Attempts to look beyond nonsmooth theory into the effects of regularization, introducing a non-ideal switch that is smooth, noisy, delayed, or hysteretic, have begun in [18, 13, 14, 12]. Finally, while attempts to explore its applications in electronics or mechanics have so far been somewhat unsatisfactory, hints of a deeper role in phase randomization can be found in [15].

To summarize the story so far, we must begin, of course, with . . .

Definition 1 A *two-fold* is a point \mathbf{x}_p in a system

$$(1) \quad \dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) > 0 \\ \mathbf{f}^-(\mathbf{x}) & \text{if } \sigma(\mathbf{x}) < 0 \end{cases} \quad \text{where} \quad \left. \begin{matrix} \sigma(\mathbf{x}_p) \\ \mathbf{f}^\pm(\mathbf{x}_p) \cdot \nabla \sigma(\mathbf{x}_p) \end{matrix} \right\} = 0,$$

and with certain non-degeneracy conditions satisfied at \mathbf{x}_p , namely $(\mathbf{f}^\pm \cdot \nabla)^2 \sigma \neq 0$, $0 \notin \mathbf{f}^\lambda \cdot \nabla \mathbf{x}$, and with transversality of the surfaces $\sigma = 0$, $\mathbf{f}^+ \cdot \nabla \sigma = 0$, $\mathbf{f}^- \cdot \nabla \sigma = 0$. We will introduce the combination \mathbf{f}^λ below.

The local dynamics depends entirely on two parameters evaluated at \mathbf{x}_p ,

$$(2) \quad \nu^+ = \frac{(\mathbf{f}^+ \cdot \nabla)(\mathbf{f}^- \cdot \nabla) \sigma}{\sqrt{|(\mathbf{f}^+ \cdot \nabla)^2 \sigma (\mathbf{f}^- \cdot \nabla)^2 \sigma|}} \quad \& \quad \nu^- = \frac{-\mathbf{f}^- \cdot \nabla \mathbf{f}^+ \cdot \nabla \sigma}{\sqrt{|(\mathbf{f}^+ \cdot \nabla)^2 \sigma (\mathbf{f}^- \cdot \nabla)^2 \sigma|}},$$

characterizing the local curvature of the flow. The product $\nu^+ \nu^-$ has a simple geometrical interpretation: it quantifies the jump in the vector field between \mathbf{f}^\pm at the singularity.

Measuring angles from to the ‘+’ or ‘−’ folds respectively, letting $s^\pm = \text{sign}(\mathbf{f}^\pm \cdot \nabla)^2 \sigma$,

$$(3) \quad \nu^+ \nu^- = -s^+ s^- \frac{\cot \phi - \cot \theta_+^+}{\cot \phi - \cot \theta_+^-} = -s^+ s^- \frac{\cot \phi + \cot \theta_-^-}{\cot \phi + \cot \theta_-^+},$$

where ϕ is the angle between the folds, and θ_j^i is the angle of \mathbf{f}^i from the ‘ j ’ fold, measured in the plane spanned by \mathbf{f}^+ and \mathbf{f}^- , with i and j denoting the labels + or −.

The leading order expansion of the two-fold singularity (sometimes called the ‘normal form’ in a somewhat loose usage of the terminology) is given [7, 4] by

$$(4) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \begin{cases} (-x_2, -s^+, \nu^+) & \text{if } x_1 > 0 \\ (x_3, \nu^-, s^-) & \text{if } x_1 < 0 \end{cases} + (O(|\mathbf{x}|^2), O(|\mathbf{x}|), O(|\mathbf{x}|)),$$

where $s^\pm = \text{sign}[(\mathbf{f}^\pm \cdot \nabla)^2 \sigma(\mathbf{x}_p)]$, and in higher dimensions $\dot{x}_{i \geq 4} = O(|\mathbf{x}|)$ for $i = 4, 5, \dots$

Bifurcation diagrams

Almost everything we understood until the year 2009 could already be found in Filippov’s book [7], but much of it was presented in the form of unexplained diagrams whose original source is unknown (with their description emerging across [17, 4, 6]).

The wealth of information we have on the leading order dynamics (the truncation of (4)) is summarized in the figure below, see [4, 6] for detail.

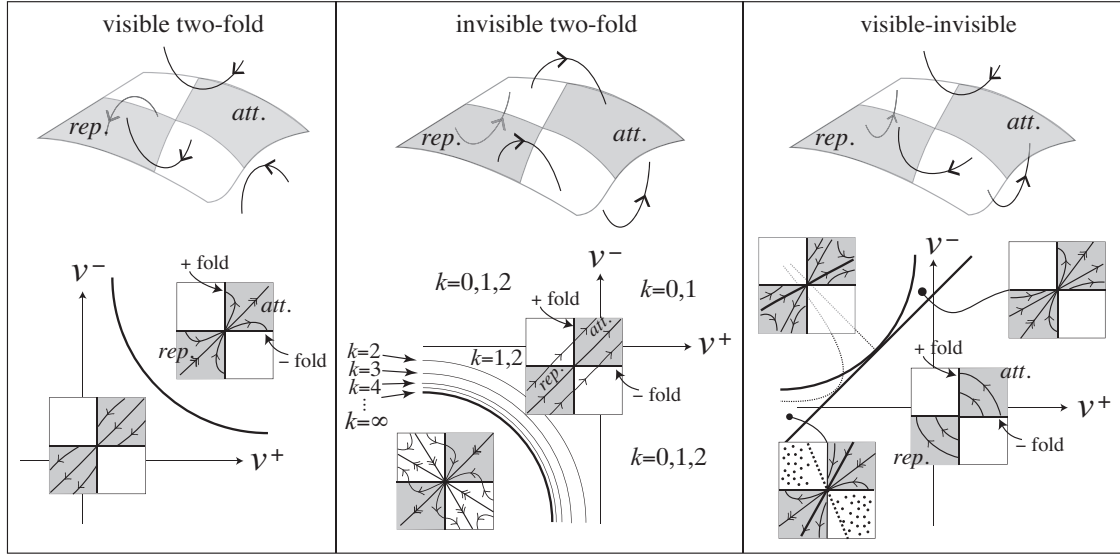


FIGURE 1. Two-folds come in three flavours, formed by the different combinations of visible or invisible folds as determined by the signs of s^\pm . Top: Regions of attracting sliding (*att.*, shaded), repelling sliding (*rep.*, shaded), and crossing (unshaded) all meet at the singularity. Bottom: Their sliding and crossing topologies in the ν^\pm parameter plane are shown below; for the invisible two-fold, k is the number of windings between visits to the sliding regions, tending to infinity where $\nu^+ \nu^- \geq 1$ in $\nu^\pm < 0$. See [4, 6] for detail.

The folds are:

- both *visible* if $s^+ > 0$ and $s^- < 0$ at \mathbf{x}_p ,
- both *invisible* if $s^+ < 0$ and $s^- > 0$ at \mathbf{x}_p ,
- one *visible* and one *invisible* if $s^+ s^- > 0$ at \mathbf{x}_p ,

(we sometimes refer to these as the *flavours* of two-fold).

Crossing maps and winding numbers

The distinguishing feature of the invisible two-fold is that the flow can wind repeatedly around the singularity, making repeated visits to the crossing regions, possibly between entry/exit points to/from the attracting/repelling sliding regions.

Let $\mathbf{y} = (x_2, x_3)$ denote a point on the switching surface $x_1 = 0$, and \mathbf{y}_i denote an iterate of the return map to the switching surface under the flow. A single return to the surface is given by

$$(5) \quad \mathbf{y}_{2m+1} = \underline{\underline{B}}^\pm \mathbf{y}_{2m}, \quad \underline{\underline{B}}^+ = \begin{pmatrix} -1 & 0 \\ -2\nu^+ & 1 \end{pmatrix} \quad \& \quad \underline{\underline{B}}^- = \begin{pmatrix} 1 & -2\nu^- \\ 0 & -1 \end{pmatrix},$$

where $\underline{\underline{B}}^+$ and $\underline{\underline{B}}^-$ are applied in $x_2 < 0$ and $x_3 < 0$ respectively. The second return map, on $x_2 < 0$ or $x_3 < 0$, is therefore

$$(6) \quad \mathbf{y}_{2m+2} = \underline{\underline{A}}^\pm \mathbf{y}_{2m}, \quad \underline{\underline{A}}^\pm = \underline{\underline{B}}^\mp \underline{\underline{B}}^\pm.$$

Because the maps are associated with folds, they are involutions, so $(\underline{\underline{B}}^+)^2 = (\underline{\underline{B}}^-)^2 = \underline{\underline{1}}$ and $\underline{\underline{A}}^+ = (\underline{\underline{A}}^-)^{-1}$. The solutions to the difference equation (6) are now obviously

$$(7) \quad \mathbf{y}_{2m} = (\underline{\underline{A}}^+)^m \mathbf{y}_0 \quad \text{or} \quad \mathbf{y}_{2m} = (\underline{\underline{A}}^-)^m \mathbf{y}_0,$$

and a little trigonometry using the substitution $\nu^+ \nu^- = \cos^2 \Theta$ provides

$$(8) \quad (\underline{\underline{A}}^\pm)^m = \frac{\sin[2m\Theta]}{\sin 2\Theta} \underline{\underline{A}}^\pm - \frac{\sin[2(m-1)\Theta]}{\sin 2\Theta} \underline{\underline{1}}.$$

This is also the source of the crossing numbers k in the previous figure. The main dynamical features revealed by the map are shown in the figure below.

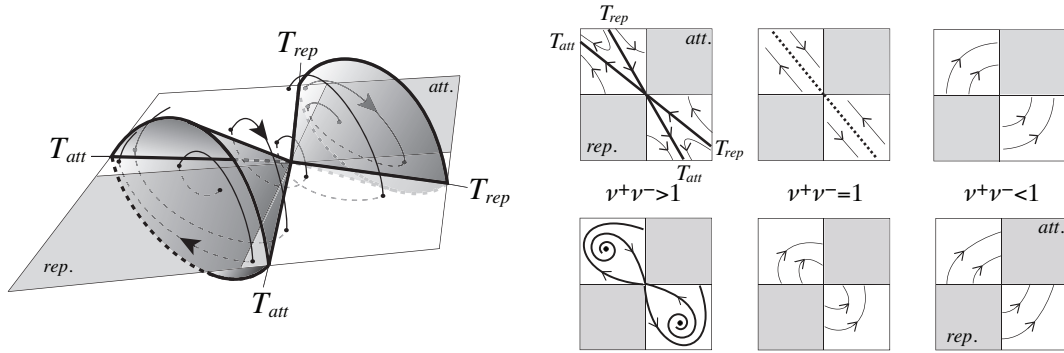


FIGURE 2. The nonsmooth diaboloid: invariant manifold (left) around an invisible two-fold. Right top: shown in the switching plane, the manifold bifurcates and disappears at $\nu^+ \nu^- = 1$, see [11]. Right bottom: the effect of higher order terms, showing a particular case leading to a determinacy-breaking attractor — as the flow exits the repelling sliding region, the crossing flow wraps it back around (via k windings) into the attracting sliding region, whereupon the sliding flow re-injects it back into the repelling region; when all local trajectories pass through the singularity, determinacy is broken; see [3].

Sliding dynamics and hidden instability

To derive sliding dynamics we need to define a *combination* of \mathbf{f}^\pm on the switching surface. It turns out that Filippov's combination hides a structural instability, in

$$(9) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1}{2} (1 + \lambda) (-x_2, -s^+, \nu^+) + \frac{1}{2} (1 - \lambda) (x_3, \nu^-, s^-),$$

essentially because the value $\lambda = \frac{x_3 - x_2}{x_3 + x_2}$ for which sliding occurs is singular at $x_2 = x_3 = 0$.

It is shown in [9] that a structurally stable combination is

$$(10) \quad (\dot{x}_1, \dot{x}_2, \dot{x}_3) = \frac{1}{2}(1 + \lambda)(-x_2, -s^+, \nu^+) + \frac{1}{2}(1 - \lambda)(x_3, \nu^-, s^-) + (1 - \lambda^2)(\alpha, 0, 0)$$

for small $\alpha \neq 0$. A well-defined manifold \mathcal{M} of sliding solutions then exists,

$$(11) \quad \mathcal{M} = \left\{ (\lambda, x_2, x_3) : \frac{1}{2}(1 - \lambda)x_3 - \frac{1}{2}(1 + \lambda)x_2 + \alpha(1 - \lambda^2) = 0 \right\},$$

inside the layer $(\lambda, x_2, x_3) \in (-1, +1) \times \mathbb{R}^2$, with \mathcal{M} normally hyperbolic except on

$$(12) \quad \mathcal{L} = \left\{ (\lambda, x_2, x_3) \subset \mathcal{M} : \lambda = 2 \frac{2\alpha + x_3 - x_2}{x_3 + x_2} = -\frac{x_3 + x_2}{4\alpha} \right\},$$

which corresponds to the two-fold magnified inside the switching layer $\lambda \in (-1, +1)$, $(x_2, x_3) \in \mathbb{R}^2$. The dynamics inside the layer is given by

$$(13) \quad (\varepsilon \dot{\lambda}, \dot{x}_2, \dot{x}_3) = \frac{1}{2}(1 + \lambda)(-x_2, -s^+, \nu^+) + \frac{1}{2}(1 - \lambda)(x_3, \nu^-, s^-) + (1 - \lambda^2)(\alpha, 0, 0),$$

for $\varepsilon \rightarrow 0$, which can be transformed into the well-known singularity of folded slow-manifolds associated with canards in smooth slow-fast systems,

$$(\varepsilon \dot{x}, \dot{y}, \dot{z}) = (y + x^2, pz + qx, r) + (\mathcal{O}(\varepsilon x, \varepsilon z, xz), \mathcal{O}(z^2, xz), \mathcal{O}(z, x))$$

provided $\alpha \neq 0$, where p, q, r , are real constants, and provided the conditions $\frac{1}{2}(\nu^+ - \nu^-) \leq 1 = -s^+ = s^-$ or $\frac{1}{2}(\nu^+ - \nu^-) \geq -1 = -s^+ = s^-$ do not hold.

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Abstract: It took nearly 30 years from the translation of Filippov's seminal book to be able to say that the two-fold singularity is understood. We now know that its structural stability requires nonlinear switching or *hidden* terms, and that it comes in three main flavours, with numerous subclasses between which bifurcations can occur. We know that it is neither an attractor nor a repeller, but a bridge between attracting and repelling sliding, and in certain cases is a source of determinacy-breaking.